Theory of Measurable Correspondences and Calculus of Variation

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by

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I. Introduction

The purpose of this paper is to summarize, in a unified and systematic way, a part of my recent contributions to the existence theory for several variational problems arising in economic analysis. In particular, the Aumann-Perles' problem, the Arkin-Levin's problem and the optimal economic growth problem are our main concerns in this paper. Detailed proofs will be omitted, and the presentations will be made as simply as possible. The readers can find more rigorous and general treatments of the problem as well as detailed proofs in my related articles or my monograph ([9] ~ [17]).

II. Measurable Correspondences

The existence theorem of measurable selections for certain correspondences (= multi-valued mappings), the first successful proof of which was obtained by J.von Neumann, has gradually been acquiring a wider range of applications. The proof of this theorem is based upon a deep insight into the topological properties of Polish or Souslin spaces. Although the idea involved in the proof deserves close attentions on its own, the theorem also provides indispensable foundations for certain problems in functional analysis, optimization theory and probability theory. (For detailed analysis, see Castaing-Valadier [6] and Maruyama [13], Chapter 6.)

1. Measurable Selections

<u>DEFINITION</u> Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two measurable spaces and consider a correspondence $\Gamma : X \twoheadrightarrow Y$. If there exists a measurable (single-valued) mapping $\gamma : X \to Y$ such that

$$\gamma(x) \in \Gamma(x) \quad \text{for all } x \in X,$$

then γ is called a *measurable selection* of Γ.

Let Y be a topological space and $\mathcal{B}(Y)$ the Borel σ-field on Y. If there exists a sequence of measurable selections $\gamma_n : X \to Y$ (n=1,2,...) such that

$$\text{C1.} \quad \overline{\{\gamma_n(x);\ n=1,2,\ldots\}} = \Gamma(x) \quad \text{for all } x \in X,$$

then $\{\gamma_n\}$ is called a *Castaing representation* of Γ.

<u>DEFINITION</u> Let $(X, \mathcal{E})$ be a measurable space and Y a topological space. A correspondence $\Gamma : X \twoheadrightarrow Y$ is said to be *measurable* if $\Gamma^{-w}(U) \in \mathcal{E}$ for every open set U in Y, where $\Gamma^{-w}(U) = \{x \in X \mid \Gamma(x) \cap U \neq \emptyset\}$.

[A] *Let* $(X, \mathcal{E})$ *be a measurable space and* Y *a Polish space. Then the following two statements are equivalent for any closed-valued correspondence* $\Gamma : X \twoheadrightarrow Y$.

(i) Γ *is measurable.*

(ii) Γ *has a Castaing representation.*

<u>DEFINITION</u> Let μ be a positive finite measure on a measurable space $(X, \mathcal{E})$. We designate by $\mathcal{E}_\mu$ the μ-completion of $\mathcal{E}$. The σ-field $\hat{\mathcal{E}}$ defined by

$$\hat{\mathcal{E}} \equiv \bigcap_{\mu} \{\mathcal{E}_\mu \mid \mu \text{ is a positive finite measure on } (X, \mathcal{E})\}$$

is called the *universal completion* of $\mathcal{E}$. If $\mathcal{E} = \hat{\mathcal{E}}$, then $\mathcal{E}$ is said to be *universally complete.*

[B] *Let* $(X, \mathcal{E})$ *be a universally complete measurable space and* Y *a Souslin space. If the graph* $G(\Gamma)$ *of the correspondence* $\Gamma : X \twoheadrightarrow Y$ *belongs to* $\mathcal{E} \otimes \mathcal{B}(Y)$, *then* Γ *has a* $(\mathcal{E}, \mathcal{B}(Y))$ *- measurable selection.*

<u>REMARK</u> Let $(X, \mathcal{E})$ be a measurable space and Y a Polish space. If $\Gamma : X \twoheadrightarrow Y$ is a closed-valued measurable correspondence, then

$$G(\Gamma) \in \mathcal{E} \otimes \mathcal{B}(Y).$$

2. Filippov's Implicit Function Theorem

[C] *Let* $(X, \mathcal{E}_X)$ *and* $(Z, \mathcal{E}_Z)$ *be measurable spaces and let* Y *be* *a Souslin space. Assume the following* (i) *and* (ii) .

(i) *The correspondences* $\Gamma : X \twoheadrightarrow Y$ *and* $\Sigma : X \twoheadrightarrow Z$ *satisfy*

$$G(\Gamma) \in \mathcal{E}_X \otimes \mathcal{B}(Y), \; G(\Sigma) \in \mathcal{E}_X \otimes \mathcal{E}_Z.$$

(ii) *The mapping* $g : X \times Y \to Z$ *is* $(\mathcal{E}_X \otimes \mathcal{B}(Y), \mathcal{E}_Z)$ - *measurable and satisfies*

$$g(x, \Gamma(x)) \cap \Sigma(x) \neq \phi \quad \textit{for all } x \in X.$$

Then there exists a $(\hat{\mathcal{E}}_X, \mathcal{B}(Y))$ - *measurable selection* γ *of* Γ *such that*

$$g(x, \gamma(x)) \in \Sigma(x) \quad \textit{for all } x \in X,$$

where $\hat{\mathcal{E}}_X$ *stands for the universal completion of* $\mathcal{E}_X$.

3. Integration of Correspondences

<u>DEFINITION</u> Let $(X, \mathcal{E}, \mu)$ be a measure space and let $\Gamma : X \twoheadrightarrow R^n$ be a measurable correspondence. Then we define the *Aumann-integral* of Γ by

$$\int_X \Gamma d\mu = \{ \int_X \gamma d\mu \mid \gamma \text{ is an integrable selection of } \Gamma\}.$$

<u>DEFINITION</u> Let $(X, \mathcal{E}, \mu)$ be a measure space. We designate by $\mathcal{F}_\Gamma$ the set of all the measurable selections of a measurable

correspondence $\Gamma : X \twoheadrightarrow R^n$. Γ is said to be *integrably bounded* if there exists an integrable function $\psi : X \to R$ such that

$$\sup_{\gamma \in \mathcal{F}_\Gamma} \| \gamma(x) \| \leqq \psi(x) \text{ a.e.}$$

Using this concept, we are able to give a striking condition which assures the compactness of the integral. The basic idea behind the following Theorem E is motivated by the Dunford-Pettis' Theorem (c.f. Maruyama [13] pp.378-380).

[E] *If* $(X, \mathcal{E}, \mu)$ *is a finite measure space and* $\Gamma : X \twoheadrightarrow R^n$ *is a compact-convex-valued measurable correspondence which is integrably bounded, then the integral of* Γ *is compact in* R^n.

The following two theorems can be derived from the Ljapunov's convexity theorem concerning to the range of a non-atomic finite dimensional vector measure.

[F] *If* $(X, \mathcal{E}, \mu)$ *is a finite, non-atomic measure space and* $\Gamma : X \twoheadrightarrow R^n$ *is measurable, then the integral of* Γ *is convex.*

[G] *If* $(X, \mathcal{E}, \mu)$ *is a finite, non-atomic measure space and* $\Gamma : X \twoheadrightarrow R^n$ *is a compact-valued measurable correspondence which is integrably bounded, then*

$$\int_X \Gamma d\mu = \int_X \ddot{\Gamma} d\mu = \int_X \text{co } \Gamma d\mu,$$

where $\ddot{\Gamma}$ *is the correspondence which associates to each* $x \in X$ *the profile of* $\Gamma(x)$.

In order to prove [F] and [G], we have to make use of the following crucial fact.

Let $\theta : X \to R^n$ be any (single-valued) integrable mapping. And define an R^n-valued measure γ by

$$\gamma(E) = \int_E \theta(x)d\mu$$

for each $E \in \mathcal{E}$. It can be proved that *the range* $\gamma(\mathcal{E})$ *of* γ *is convex provided that* μ *is finite and non-atomic.*

III. Aumann-Perles' Problem

Let us begin by the simplest problem which frequently appears in mathematical economics or game theory.

First, the following items are assumed to be given.

$$u : [0, 1] \times R^l \to R,$$

$$g : [0, 1] \times R^l \to R^l,$$

$$\Gamma : [0, 1] \twoheadrightarrow R^l,$$

$$\omega \in R^l.$$

And consider the problem:

$$
(\mathrm{A}-\mathrm{P})\left\{
\begin{array}{lr}
\underset{x}{\mathit{Maximize}} \displaystyle\int_0^1 u(\mathrm{t}, x(\mathrm{t}))\, \mathrm{dt} & (1) \\[2ex]
\quad \mathit{subject\ to} & \\[2ex]
\qquad \displaystyle\int_0^1 g(\mathrm{t}, x(\mathrm{t}))\, \mathrm{dt} \leqq \omega & (2) \\[2ex]
x \in F_\Gamma;\ \text{i.e. } x \text{ is a measurable selection of } \Gamma & (3)
\end{array}
\right.
$$

(dt denotes the Lebesgue measure.)

We shall assume the following conditions.

<1> u and g are Carathéodory mappings; i.e. both of these mappings are measurable on [0, 1] and continuous on R^n separately.

<2> Γ is a compact-valued measurable correspondence.

<3>
$$\tilde{u}(\mathrm{t}) = \sup_{\mathrm{x} \in \Gamma(\mathrm{t})} | u(\mathrm{t}, \mathrm{x}) | < \infty$$

$$\tilde{g}(\mathrm{t}) = \sup_{\mathrm{x} \in \Gamma(\mathrm{t})} \| g(\mathrm{t}, \mathrm{x}) \| < \infty$$

are integrable.

Our aim is to show the existence of a measurable mapping $x^* : [0, 1] \to R$ which maximizes the integral functional (1) under the constraints (2) and (3).

Define a mapping $f : [0, 1] \times R^{l} \to R^{l+1}$ by

$$f(t, x) = (u(t, x) , g(t, x)).$$

Then f is a Caratheodory mapping because of the assumption <1>. If we define the correspondence $\Lambda : [0, 1] \twoheadrightarrow R^{l+1}$ by

$$\Lambda(t) = f(t, \Gamma(t)),$$

then Λ is compact-valued and measurable. It should be noted that

$$F_{\Lambda} = \{ f(t, x(t)) \mid x \in F_{\Gamma} \}$$

according to the Filippov's implicit function theorem (I-[C]).
By the definition of the Aumann-integral,

$$\begin{aligned}\int_0^1 \Lambda\, dt &= \{ \int_0^1 \lambda(t)dt \mid \lambda \in F_{\Lambda} \} \\ &= \{ \int_0^1 f(t, x(t))dt \mid x \in F_{\Gamma} \} \\ &= \{ (\int_0^1 u(t, x(t))dt, \int_0^1 g(t, x(t))dt) \mid x \in F_{\Gamma} \}.\end{aligned}$$

This is the set of all the combinations of

$$\int_0^1 u(t, x(t))dt \quad \text{and} \quad \int_0^1 g(t, x(t))dt$$

such that x satisfies the constraint (3).

Since Λ is integrably bounded by <3>, we must have

$$\int_0^1 \Lambda(t)dt = \int_0^1 \text{co } \Lambda(t)\, dt$$

and this integral is compact and convex in R^{l+1} by I-[G].

However, this set is too big because it may contain those elements which do not satisfy the constraint (2). Thus the set of all the really admissible combinations is given by

$$K \equiv \{ (a, b) \in R \times R^{l} \mid (a, b) \in \int_0^1 \Lambda(t)dt,\ b \leqq \omega \}.$$

Since K is clearly compact in R^{l+1}, its projection on the first axis is also compact. Hence it has the greatest element a*. Let b* be any element in R^{l} such that

$$(a^*, b^*) \in \int_0^1 \Lambda(t)dt.$$

Then by the definition of the Aumann-integral, there exists some $x^* \in F_{\Gamma}$ such that

$$\begin{aligned}(a^*, b^*) &= \int_0^1 f(t, x^*(t))dt \\ &= \left(\int_0^1 u(t, x^*(t))dt,\ \int_0^1 g(t, x^*(t))dt\right).\end{aligned}$$

Clearly, this x^* is an optimal solution for the problem (A - P).

The existence conditions for the variational problem of this kind have been studied by Arkin-Levin [1], Artstein [2], Aumann-Perles [3] and Berliocchi-Lasry [5]. Furthermore, Maruyama [9], [12] examined a somewhat sophisticated problem as follows.

$$\underset{x,\ E}{\textit{Maximize}} \int_E u(t, x(t))dt$$

subject to

$$\int_E g(t, x(t))dt \leqq \omega$$

x is measurable selection of Γ

E is a measurable set in [0, 1].

In order to solve this problem in a general space, the disintegration theory of Radon measures was effectively used. (See also Maruyama [10].)

IV. Arkin-Levin's Problem

The following four mappings are assumed to be given.

$u : [0, 1]^2 \times R^l \to R,$

$g : [0, 1]^2 \times R^l \to R^l,$

$\Gamma : [0, 1] \twoheadrightarrow R^l,$

$\omega : [0, 1] \to R^l.$

And consider the problem:

$$(A - L)\begin{cases} \textit{Maximize} \displaystyle\int_0^1 \int_0^1 u(s,\ t,\ x(s,\ t))dsdt & (1) \\ \textit{subject to} & \\ \displaystyle\int_0^1 g(s,\ t,\ x(s,\ t))ds \leqq \omega(t) & (2) \\ x \in F_\Gamma & \end{cases}$$

(ds and dt denote the Lebesgue measures.)

We shall assume the following conditions.

<1> u and g are Carathéodory mappings; i.e. both of these mappings are measurable on $[0,\ 1]^2$ and continuous on R^l separately.

<2> Γ is a compact-valued measurable correspondence.

<3> $$\tilde{u}(s,\ t) = \sup_{x \in \Gamma(s,t)} |\ u(s,\ t,\ x)\ | < +\infty$$

$$\tilde{g}(s,\ t) = \sup_{x \in \Gamma(s,t)} \|\ g(s,\ t,\ x)\ \| < +\infty$$

are integrable.

Define a mapping $f : [0,\ 1]^2 \times R^l \to R \times R^l$ by

$$f(s,\ t,\ x) = (u(s,\ t,\ x)\ ,\ g(s,\ t,\ x)).$$

Then f is a Carathéodory mapping because of the Assumption <1>.
If we define

$$\Lambda(s, t) = f(s, t, \Gamma(s, t)),$$

then Λ is a compact-valued measurable correspondence. It should be noted that

$$F_\Lambda = \{ f(s, t, x(s, t)) \mid x \in F_\Gamma \}$$

according to the Filippov's implicit function theorem (I-[C]).

Since the correspondence Λ may not be convex-valued, it is sometimes very convenient to consider the correspondence $\Delta : [0, 1]^2 \twoheadrightarrow R^{l+1}$ defined by

$$\Delta(s, t) = \mathrm{co}\, \Lambda(s, t).$$

Δ is clearly compact-convex-valued measurable correspondence. Since Λ is integrably bounded by <3>, we can prove one of our crucial results as follows.

[A] F_Δ *is weakly compact and convex.*

Define a bounded linear operator H on $L^1([0, 1]^2, R^{l+1})$ into $R \times L^1([0, 1], R^l)$ by

$$H : (\alpha(s, t), \beta(s, t)) \to \left(\int_0^1 \int_0^1 \alpha(s, t)\, ds\, dt, \int_0^1 \beta(s, t)\, ds\right)$$

where $\alpha \in L^1([0, 1]^2, R)$ and $\beta \in L^1([0, 1]^2, R^l)$.

[B] $H(\mathcal{F}_\Delta)$ *is convex and weakly compact.*

Since $\mathcal{F}_\Lambda \subset \mathcal{F}_\Delta$, it is apparent that

$$H(\mathcal{F}_\Lambda) \subset H(\mathcal{F}_\Delta).$$

However the converse inclusion, which is not obvious, is also true, and we can establish the following result.

[C] $H(\mathcal{F}_\Delta) = H(\mathcal{F}_\Lambda)$.

In order to establish this result, we effectively make use of the infinite dimensional version of the Ljapunov's convexity theorem concerning the range of non-atomic vector measures. Let me explain more concretely. Suppose that

$$\theta_1 : [0, 1]^2 \to R$$

$$\theta_2 : [0, 1]^2 \to R^l$$

are any integrable mappings. And define

$$\gamma(E) = \left(\int_E \theta_1 ds dt, \int_E \theta_2 ds \right)$$

for each measurable set $E \subset [0, 1]^2$. Then γ is an $R \times L^1([0, 1], R^l)$ -valued measure. Through a somewhat complicated argument, we can prove that the range of γ is convex. This result is the key in the proof of [C]. (c.f. Maruyama [16].)

$H(F_\Lambda)$ represents the set of all the combination of

$$\int_0^1\int_0^1 u(s, t, x(s, t))\, ds\, dt \quad \text{and} \quad \int_0^1 g(s, t, x(s, t))\, ds$$

taking account of the constraint (3). But the set $H(F_\Lambda)$ is too big because it may contain those elements which do not satisfy the constraint (2). Thus the set of all the really admissible combinations is given by

$$K = \{(a, b(t)) \in H(F_\Lambda) \mid b(t) \leqq \omega(t) \text{ a.e.}\}.$$

Since K is weakly compact and convex, its projection on R is compact and hence it has the greatest element a*. Pick up any $b^*(t) \in L^1([0, 1], R^l)$ such that

$$(a^*, b^*(t)) \in K.$$

Then there exists a measurable selection λ^* of Λ such that

$$H(\lambda^*) = (a^*, b^*(t)).$$

Finally, thanks to the Filippov's measurable implicit function theorem, we can find out a measurable mapping $x^* : [0, 1]^2 \to R^l$ such that

$$f(s, t, x^*(s, t)) = \lambda^*(s, t)$$

$$x^*(s, t) \in \Gamma(s, t) \quad \text{a.e.}$$

Clearly x^* is an optimal solution of our problem.

In 1972, V.I.Arkin and V.L.Levin, excellent Russian mathematicians in this field, rigorously proved the existence of optimal solutions for a similar variational problem. Maruyama [11], [16] recapitulated this problem in a much more general setting and gave an existence proof through a modern reformulation of the Arkin-Levin's method.

V. Optimal Economic Growth

First a couple of mappings u and f is assumed to be given.

$$u : R_+ \times R_+ \to R_+ \text{ , } u(t, x) \text{ ,}$$

$$f : R_+ \times R_+ \to R_+ \text{ , } f(t, k) \text{ .}$$

Here $u(t, x)$ is interpreted as the utility of some representative economic agent at time t when his consumption is x, and $f(t, k)$ is interpreted as an output produced at time t when the quantity of capital stock is k.

Furthermore we have a couple of variable mappings to be optimized.

$$k : R_+ \to R_+ , \; k(t),$$

$$s : R_+ \to [0, 1] , \; s(t).$$

$k(t)$ is the quantity of capital stock at time t, and $s(t)$ is the saving rate at time t.

Then our problem is formulated as follows:

Maximize

$$J(k, s) = \int_0^\infty u[t, (1 - s(t)) f(t, k(t))] e^{-\delta t} dt$$

$$\equiv \int_0^\infty w(t, k(t), s(t)) e^{-\delta t} dt$$

subject to

$$\dot{k}(t) = s(t) f(t, k(t)) - \lambda k(t)$$

$$\equiv g(t, k(t), s(t))$$

$$k(0) = \bar{k}.$$

Here δ is the discount rate for the utility in the future, and λ is the depreciation rate of capital stock. Both of δ and λ are assumed to be positive.

Since the underlying measure space is $[0,\infty)$, it is naturally anticipated that some technical difficulties will arise. In order to avoid such difficulties, we introduce a finite measure ν on R_+

defined by

$$\nu(E) = \int_E e^{-\delta t}\, dt. \qquad \text{for every measurable set E.}$$

Then our problem can be rewritten in the following form:

Maximize

$$J(k, s) = \int_0^\infty w(t, k(t), s(t))\, d\nu \tag{1}$$

subject to

$$\dot{k}(t) = g(t, k(t), s(t)) \tag{2}$$

$$k(0) = \overline{k}. \tag{3}$$

We denote by S the set of all the measurable functions $s : R \to [0, 1]$, and by $W_\delta^{1,2}$ the weighted Sobolev space with the weight function $e^{-\delta t}$. A pair (k, s) in $W_\delta^{1,2} \times S$ is said to be an *admissible pair* if it satisfies (2) and (3). The set of all the admissible pairs is denoted by A and its projection on $W_\delta^{1,2}$ is denoted by A_k.

We shall assume the following conditions.

<1> u is measurable in (t, x). Furthermore u is upper semi-continuous and concave in x.

<2> f is continuous in (t, k).

<3> There exists some positive number C > 0 such that

$k \geqq C$ implies $\sup_{t \in R_+} f(t, k) \leqq \lambda k.$

<4> There exists a couple of positive constants α and β such that

$$0 < \beta < \delta/2$$

$$f(t, k) < \alpha\ |k|\ e^{\beta t}$$

for all $t \in R_+$ and $k \in R_+$.

<5> There exists a non-negative ν-integrable function $\theta : R_+ \to R$ and a number b such that

$$w(t, k, s) - bg(t, k, s) \leqq \theta(t)$$

for every (t, k, s).

Thanks to our Assumption <3> and <4>, we can prove that

[A] *A_k is weakly sequentially compact in $W_\delta^{1,2}$.*

If we take account of Assumption <5> together,

[B] $\gamma = \sup_{(k,\ s) \in A} J(k, s) < \infty.$

Let $\{(k_n, s_n)\}$ be a sequence in A such that

$$\lim_{n} J(k_n, s_n) = \gamma.$$

Since A_k is weakly sequentially compact by [A], $\{k_n\}$ has a weakly convergent subsequence. Hence we can assume, without loss of generality, that

$$k_n \to k^* \quad \text{weakly in } W_\delta^{1,2} .$$

The most important and difficult step in the proof is to show the following proposition.

[C] *There exists an* ν*-integrable function* $\zeta : R_+ \to R$ *such that*

(i) $$\int_0^\infty \zeta(t)\, d\nu \geqq \gamma$$

(ii) *for every* $t \in R_+$, *there exists* $s^*(t) \in [0, 1]$ *such that*

$$\dot{k}(t) = g(t, k^*(t), s^*(t))$$

$$0 \leqq \zeta(t) \leqq w(t, k^*(t), s^*(t)) .$$

($s^*(t)$ *may not be measurable so far.*)

Finally we have to show that we can choose $s^*(t)$ so as to be measurable. In order to achieve this object, the Filippov's implicit function theorem is quite effective. Clearly ($k^*(t)$, $s^*(t)$) is an optimal pair in A.

The detailed analysis can be found in Maruyama [14], [15], [17]. The author is indebted to Berkovitz [4] and Chichilniski [7] for various ideas embodied in the proof.

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